Practice Exam Percolation Theory

(Real exam will be on : 4 November 2022, 8:30-10:30)

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (20 pts).
State and prove the BK inequality.

Exercise 2 (20 pts).
Let a_n denote the number of (self-avoiding) paths in $\mathbb{Z}^d$ starting from $\underline{0}$.

i) Fekete's lemmas states that if a sequence of nonnegative numbers $(b_n)_n$ satisfies $b_{n+m} \leq b_n + b_m$ for all $n, m \in \mathbb{N}$ then $\lim_{n\to\infty} \frac{b_n}{n}$ exists and equals $\inf_n \frac{b_n}{n}$.

Prove Fekete's lemma.

ii) The *connective constant* $\gamma(\mathbb{Z}^d)$ of $\mathbb{Z}^d$ is defined as $\lim_n \frac{1}{n} \ln a_n$. Show that this limit exists.

Exercise 3 (25 pts)

i) Show that if $f : \{\pm 1\}^n \to \{\pm 1\}$ satisfies $\sum_{|S|=1} \hat{f}^2(S) = 1$ then $f(x) = \pm x_i$ for some i.

ii) Prove the "dictators are stablest theorem" : for all $0 < \rho < 1$ and all $f : \{\pm 1\}^n \to \{\pm 1\}$ satisfying $\mathbb{P}(f = 1) = \mathbb{P}(f = -1) = 1/2$ (i.e. f corresponds to a "fair" election system) we have $\mathrm{Stab}_\rho(f) \leq \rho$ with equality if and only if $f = \pm x_i$ for some i.

Exercise 4 (25 pts)
We consider the following *directed* percolation model on $\mathbb{Z}^2$. Every edge is randomly replaced with an arrow either pointing to the right or left for horizontal edges, and either pointing up or down for vertical edges. The edges independently chose such an orientation. Left, respectively up, has probability $p \in [0, 1]$. We say percolation occurs if there is an infinite path following the arrows. (Note it does not have to by "bi-infinite". I.e. it does not have to extend infinitely far in both directions).

i) Show that in this model $\mathbb{P}_p(\text{percolation}) \in \{0, 1\}$ for all $p \in [0, 1]$.

ii) Show that there exists an $\varepsilon > 0$ such that $\mathbb{P}_p(\text{percolation}) = 1$ for all $p < \varepsilon$ and all $p > 1 - \varepsilon$.

iii) Show that $\mathbb{P}_{1/2}(\text{percolation}) = 0$.

(The end)